A POPULATION WITH CONTINUALLY DECLINING MORTALITY

Robert Schoen

Department of Sociology
Pennsylvania State University
University Park PA 16802

Stefan Hrafn Jonsson

Icelandic Center for Social Research and Analysis and
Department of Sociology
Pennsylvania State University
University Park PA 16802

Paula Tufis

Department of Sociology
Pennsylvania State University
University Park PA 16802

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ABSTRACT

In recent years, many countries have experienced sustained declines in death rates. Here we present a new dynamic population model that provides a framework for analyzing continuing mortality declines. Every year there is one birth in the model. Mortality increases exponentially over age at rate b while decreasing exponentially over time at rate c. The model population is strikingly linear in its behavior over time, with many measures changing at a rate that closely approximates a simple function of b and c. The size of the population is virtually the same as the average age at death, and both increase annually by c/b. Period life expectancy at birth also increases linearly by c/b, while the average age of the population increases linearly by c/(2b). Preserving a constant ratio of persons in the economically active ages to those in the retirement ages implies an increase in the "normal" age of retirement of about 6.8c years per year. The interpretability of parameters b and c, the ability to accommodate varying rates of decline, and the linear nature of demographic changes enhance the model's potential for analyzing steadily increasing longevity.

A Population With Continually Declining Mortality

People today are living longer than ever before, and there is no sign that mortality reductions are coming to an end. White (2002) examined life expectancies in 21 high income countries from 1955 to 1996, and found a clear upward, linear trend. Kannisto, Lauritsen, Thatcher, and Vaupel (1994) found continuing improvements in mortality at the highest ages in 27 low mortality countries, with no reduction in the rate of decline from the 1960s through the 1980s. The number of centenarians is increasing markedly. Robine and Saito (2003) saw a strong and continuing acceleration in the emergence of the centenarian population in Japan, currently the nation with the world's highest life expectancy. Oeppen and Vaupel (2002) followed the time trend of longevity in the country with the highest life expectancy during each time period, and found that over the past 160 years the maximum female life expectancy rose at a steady pace: nearly 3 months per year. There was no sign of slowing in recent years as that maximum expectation of life at birth approached 85 years.

Those steady improvements in longevity affect more than individual survival because they impact population age structure and patterns of intergenerational transfer. In a number of countries, including the United States, the costs of medical care and retirement benefits have become matters of national concern. The relative claim of older versus younger persons to societal support continues to be a contested area of social policy (cf. Preston 1984).

Established demographic techniques, such as population projections and the life table, can accurately describe the implications of given schedules of age-specific death rates. However, analyses of the present situation of sustained mortality decline can greatly benefit from the development of new techniques (e.g. Lee and Carter 1992). Here we focus on closed form dynamic mortality models. After examining past approaches, we propose a new dynamic model of continually declining mortality and explore its demographic implications.

PREVIOUS APPROACHES TO DYNAMIC MORTALITY

A life table shows the number of persons from an initial birth cohort who survive to each age under a given schedule of age-specific mortality rates. If rates for a given year are used, a period life table is produced. If the rates follow an actual birth cohort, a cohort life table results. In either case, the stationary population of the life table describes the number and composition of the population implied by those rates (and the initial number in the life table cohort). The period and cohort manifestations of the life table reflect the two customary perspectives used in demography.

When rates are changing, there are useful model populations that do not conform to either the period or cohort perspectives. There is a need for a third perspective, first identified by Brouard (1986) and since discussed by Guillot (2003a). Consider a model population that has had one birth each year for many years, but has had a history of changing mortality. During year t, the number of persons in that population at age x is the number of survivors of the cohort born x years earlier, survivors who have experienced, at each age, the mortality rates that prevailed at an earlier time. The time t population of the model is not a “period” population in the life table sense because it is not determined by the rates prevailing in any one period. Neither is it a cohort population. It is rather what may be termed a “wedge-period” population, since the set of rates that determine the model’s survivors at time t form a triangular, wedge-like pattern in a cross-tabulation (or Lexis diagram) of the population’s mortality by age and time.

Brouard (1986) termed the number of persons in such a wedge-period population the "duree de vie moyenne actuelle". Guillot (2003a) translated that as the "cross-sectional average length of life", and denoted the model population size at time t by CAL(t). From its definition, CAL(t) is the sum of cohort proportions surviving, where the sum is taken over all of the cohorts present at time t. When mortality has been improving, CAL(t) is always less than the time t expectation of life at birth, because a higher proportion of persons survive to each age under the death rates prevailing at time t than under the rates experienced in the past. As a measure of population size, CAL shows the impact of past and current mortality on population growth (Guillot 2003a). Steadily improving mortality implies a future population increase, i.e. a mortality generated population momentum. Guillot (2003a) showed that if mortality becomes constant at its time t level, the model population will grow in size by the factor $e_0(t)/CAL(t)$, where $e_0(t)$ is the time t life expectancy.

Bongaarts and Feeney (2002), in a significant advance in modeling changing mortality, saw CAL as a tempo-adjusted measure of mortality. While that interpretation has been criticized by Vaupel (2002) and Guillot (2003ab), we focus here on the underlying Bongaarts-Feeney dynamic mortality model. They assumed that at every time the force of mortality had a Gompertz pattern, i.e. that

$$\mu(x,t) = \mu(0,t)\, e^{bx} \tag{1}$$

where $\mu(x,t)$ is the force of mortality (i.e. the intensity or instantaneous risk of death) at age x and time t and b is the fixed Gompertz parameter. Bongaarts and Feeney (2002) assumed that the *shape* of the force of mortality curve did not change, but that it could shift over time by sliding along the x-axis. In particular, they argued that if $\mu(x,t)$ was multiplied at all ages by a factor $k<1$, then the new mortality function is the old mortality function shifted to the right.

Under that pattern of mortality change, Bongaarts and Feeney (2002) found that $N(x,t)$, the number of persons in the wedge-period model population at age x and time t, is given by

$$N(x,t) = N(x-F(t), 0) \qquad \text{for } x \geq F(t) \tag{2}$$

and

$$N(x,t) = 1 \qquad \text{for } x < F(t) \tag{3}$$

where the model population is scaled to have one birth each year, and $F(t)$ is the amount the N function has shifted to the right at time t. Thus population age structure N also keeps a constant shape over time, although shifting according to translation function F. The total number of persons in the population at time t, i.e. $\Sigma_i N(i,t)$, is $CAL(t)$, and the translation function can be written

$$F(t) = CAL(t) - CAL(0) \tag{4}$$

Since $CAL(0)$ is a constant, equation (4) implies

$$dF(t)/dt = dCAL(t)/dt \tag{5}$$

After deriving those relationships, Bongaarts and Feeney (2002) found that

$$CAL(t) = A_d(t) \tag{6}$$

where $A_d(t)$ is the time t model population average age at death, and that

$$CAL(t) = e_0(t) + (1/b) \ln\{ 1 - dCAL(t)/dt \} \tag{7}$$

From equation (7), $CAL(t)$ can be found from the life expectancy function and Gompertz constant b using standard numerical methods, with $N(x,t)$ following from equations (3) and (4) and $N(x,0)$. Bongaarts and Feeney (2002) did not provide an explicit nonrecursive solution for either $CAL(t)$ or $N(x,t)$.

The Bongaarts-Feeney mortality model allows for any pattern of mortality shifts over time. The Gompertz pattern is quite reasonable, though the model's bifurcated definition of μ is analytically awkward. The model preserves a constant age structure, though it is only constant over a shifting threshold age $F(t)$.[1] Preserving the initial population composition in that manner is less than optimal because it allows the choice of initial time point to influence model values. Consider a population with cyclically fluctuating mortality. If the initial time point is when mortality is at its *maximum*, $F(t) \geq 0$ for all t. For x and $t > 0$, there are numerous ages and times where the size of the population is 1 and $\mu(x,t)=0$. In contrast, if $t=0$ when mortality is at its *minimum*, then $F(t) \leq 0$. For x and $t > 0$, it follows that $\mu(x,t)$ is always greater than 0 and that the size of the population is always less than 1.

MODELING CONTINUALLY DECLINING MORTALITY

Specifying a Model of Continuous Mortality Decline

We seek to provide a framework for analyzing sustained mortality decline. While our model roughly approximates behavior in a number of developed countries during the past several decades, our intent is not to predict future mortality trends or estimate relationships in actual populations. Instead, our goal is to provide a context in which such relationships can be analyzed, and to identify some structural consequences that follow from a constant birth sequence and falling mortality.

The Gompertz formulation, where the force of mortality increases exponentially with age, is the classic model of mortality. It is not appropriate at younger ages where death rates are declining or where accidental mortality is high, and there is mounting evidence that it overstates death rates at the highest ages. Nonetheless, the exponential curve generally fits adult mortality quite well, and no alternative provides such a simple, analytically tractable pattern (Carnes, Olshansky, and Grahn 1996; Wilmoth 1997).

For our model of continuously declining mortality, we specify a Gompertz-like force of mortality that at age x and time t is

$$\mu(x,t) = e^{A+bx-ct} \tag{8}$$

Parameter A reflects the value of $\mu(0,0)$; parameter b is the fixed rate of increase in mortality over age; and parameter c is the constant rate of mortality decrease (at all ages) over time.

In the period life table for year t, the probability of surviving from birth to age x, $p_{LT}(x,t)$, is given by

$$\begin{aligned} p_{LT}(x,t) &= \exp[\, - \int_0^x \mu(a,t)\, da \,] \\ &= \exp[\, e^{A-ct} (1 - e^{bx}) / b] \qquad (9) \end{aligned}$$

The time t period life expectancy is then

$$e_0(t) = \int_0^\infty p_{LT}(x,t)\, dx \tag{10}$$

From a cohort perspective, the probability of surviving to attain age x at time t is given by

$$\begin{aligned} p(x,t) &= \exp[\, - \int_0^x \mu(a, t-x+a)\, da \,] \\ &= \exp[\, -e^{A-ct} \{e^{bx} - e^{cx}\} /(b-c)] \qquad (11) \end{aligned}$$

when $b \neq c$. With one birth occurring in the model population each year, p(x,t) provides the number in the wedge-period population at age x and time t. The total size of the time t population is

$$CAL(t) = \int_0^\infty p(x,t)\, dx \tag{12}$$

In short, our model population, closed to migration, has one birth each year, p(x,t) persons age x at time t, and an underlying force of mortality function exp[A+bx-ct]. The model size and age composition emerge as a consequence of the constant birth sequence and the absence of "feedbacks" between fertility and mortality.

Relationships in the Continuously Declining Mortality Model

Changing Life Expectancy. To examine mortality dynamics, let us begin by considering how period life expectancy increases over time. Differentiating $e_0(t)$ using equations (9) and (10), we find

$$de_0(t)/dt = (c/b)\,[\,1 - \mu(0,t)\, e_0(t)\,] \qquad (13)$$

With low mortality characterizing our model population, $\mu(0,0)$ is on the order of .0000125. Thus if $e_0(0)$ equals 80 years, the second factor in the brackets on the right side of equation (13) is about .001. Disregarding that modest amount, we can write the discrete approximation

$$\Delta e_0(t) \approx (c/b) \qquad (14)$$

or that life expectancy increases linearly by about (c/b) years each year. Since b is typically close to 0.1, a 1% annual decline in mortality over time (i.e. c=.01) implies that life expectancy increases by about 0.1 years per year.

Changing Population Size and Crude Vital Rates. The change over time in the size of the model population can be found by differentiating equations (11) and (12). That yields

$$dCAL(t)/dt = [c/(b-c)]\,[\,\int p(x,t)\,\mu(x,t)\,dx - \int p(x,t)\,\mu(cx/b, t)\,dx\,] \qquad (15)$$

where the integrals range from 0 to ∞. [Unless otherwise indicated, all integrals used have that range.] The first integral on the right of equation (15) represents the total number of deaths in the model population at time t, and can be denoted D(t). The second integral, D#(t), is the number of deaths that would occur if the death rate at age x were given by exp[A+cx-ct], i.e. the force of mortality at age (c/b)x. With b=0.1 and c=0.01, the death rate at age 100 would then be μ([.01/.1]100,t) or μ(10,t). At t=0, those parameters yield a death rate at age 100 of about .00004, so we can generally disregard the small offset introduced by D#(t). In discrete terms, we know that

$$\Delta CAL(t) = 1 - D(t) \tag{16}$$

as the increase in size during year t equals the one birth minus the number of deaths during the year. Ignoring D# and using equation (16) in equation (15) yields

$$\Delta CAL(t) \approx (c/b) \tag{17}$$

Thus the size of the model population increases annually by, approximately, amount (c/b), the same linear increase found for life expectancy in equation (14). Since with one birth each year ΔCAL cannot be greater than 1, equation (17) is not appropriate when $c \geq b$. Instead, when $c \geq b$, the value of ΔCAL approaches 1.

The time t growth rate of the model population, RNI(t), can be approximated by disregarding D#(t) in equation (15), dividing both sides of that equation by CAL(t), and noting that the crude death rate at time t, CDR(t) = D(t)/CAL(t). The result is

$$d \ln CAL(t)/dt = RNI(t) \approx [c/(b-c)]\ CDR(t) \tag{18}$$

where ln indicates the natural logarithm function. Because RNI=CBR-CDR and CBR=1/CAL, equation (18) implies that the crude birth rate at time t, CBR(t) can be approximated by

$$CBR(t) \approx [b/(b-c)]\ CDR(t) \approx (b/c)\ RNI(t) \tag{19}$$

Annual Number of Deaths. From equations (18) and (19), the total annual number of deaths can be written

$$D(t) \approx (b-c)/b \tag{20}$$

The linear growth in the size of the population thus results from a fixed number of births and an approximately constant number of deaths.

The constancy in the number of annual deaths is a key feature of the present model. It is not immediately obvious that the changing mortality rates, when applied to a wedge-period population whose size and structure are also changing, would yield a nearly constant number. To try and appreciate why that happens, let us ignore the typically very small e^{cx} term on the right hand side of equation (11) and write

$$p^*(x,t) = \exp[\ -e^{A+bx-ct}/(b-c)\ ] = \exp[\ -\mu(x,t)/(b-c)\ ] \tag{21}$$

Equation (21) shows that the approximate wedge-period population $p^*(x,t)$ is a relatively simple function of $\mu(x,t)$—that is of a single age-time-specific force of mortality. Using $p^*(x,t)$, the approximate number of deaths at time t is

$$D^*(t) = \int \exp[-\mu(x,t)/(b-c)]\ \mu(x,t)\ dx \tag{22}$$

If $u=\mu(x,t)$, then $du=\mu(x,t)dx/b$, and that substitution allows the integral in equation (22) to be evaluated analytically. Simplifying the result, by assuming that $\mu(0,t)$ is zero, yields equation (20). The exponential relationship between p^* and μ in equation (21), and the fact that

$$(\partial/\partial x)\ p^*(x,t) = p^*(x,t)\ \mu(x,t)\ [-b/(b-c)] \tag{23}$$

causes the changes in population and mortality to offset one another over the lifespan and keep D(t) approximately constant.

Age-Specific Growth and Population Entropy. Following Preston and Coale (1982), the age-specific growth rate of the population age x at time t, r(x,t), is given by

$$r(x,t) = \partial \ln p(x,t)/\partial t = [c/(b-c)]\ [\ \mu(x,t) - \mu(cx/b,\ t)\ ] \tag{24}$$

Using equations (11) and (21), equation (24) can be rewritten as

$$r(x,t) = -c \ln p(x,t) \approx [c/(b-c)]\ \mu(x,t) \qquad (25)$$

The greater the force of mortality, or the higher the age, the faster the rate of age-specific growth. Using equation (25), the crude rate of natural increase of the model population at time t is given by

$$RNI(t) = \int p(x,t)\ r(x,t)\ dx\ /\ CAL(t) = -c \int p(x,t) \ln p(x,t)\ dx\ /\ CAL(t) \qquad (26)$$

The entropy of a population at time t, $H_p(t)$, provides a measure of the randomness (or the information content) in the age distribution at time t. The minimum value of $H_p(t)$ is 0; a large value, say 1, indicates a high degree of randomness (or little information content) in the population distribution. Following the customary manner of defining entropy in mortality analyses (cf. Keyfitz 1977; Vaupel and Canudas Romo 2003), we can write

$$H_p(t) = -\int p(x,t) \ln p(x,t)\ dx\ /\ CAL(t) \qquad (27)$$

[The minus sign is introduced because $\ln p(x,t) < 0$.] Equations (18), (26), and (27) show that

$$d \ln CAL(t)/dt = RNI(t) \approx c\ H_p(t) \qquad (28)$$

From equations (19) and (28), it follows that

$$CBR(t) \approx b\ H_p(t) \qquad (29)$$

and

$$CDR(t) \approx (b-c)\ H_p(t) \qquad (30)$$

Thus the crude rates of birth, death, and natural increase are all approximated by simple functions of entropy and parameters b and c. As equation (17) shows that the amount of increase in population size is approximately linear, equation (28) implies that $H_p(t) \to 0$ as $t \to \infty$. The three crude rates thus decline monotonically over time as entropy decreases.

Mean Population Age. The mean age of the model population at time t, $A_p(t)$, is given by

$$A_p(t) = \int x\, p(x,t)\, dx \,/\, CAL(t) \qquad (31)$$

Taking the time derivative of $A_p(t)$ yields[2]

$$dA_p(t)/dt = A_p(t)\ \{\ [\int x\, p(x,t)\, r(x,t)\, dx \,/\, \int x\, p(x,t)\, dx] - c\, H_p(t)\ \} \qquad (32)$$

Numerically, as shown below, $A_p(t)$ is close to $e_0(t)/2$, and consequently $\Delta\, A_p(t) \approx c/(2b)$. The mean age of the population increases linearly by an amount that is approximately half the size of the increase in life expectancy and population size.

Mean Age at Death. The mean age at death in the model population at time t is

$$A_d(t) = \int x\, p(x,t)\, \mu(x,t)\, dx \,/\, D(t) \qquad (33)$$

The time derivative of $A_d(t)$ gives

$$dA_d(t)/dt = A_d(t)\ \{\ [\int x\, p(x,t)\, \mu(x,t)\, r(x,t)\, dx \,/\, \int x\, p(x,t)\, \mu(x,t)\, dx] -$$

$$\int p(x,t)\, \mu(x,t)\, r(x,t)\, dx \,/\, \int p(x,t)\, \mu(x,t)\, dx]\} \qquad (34)$$

To simplify equations (33) and (34), we can use approximate population $p^*(x,t)$ from equation (20), equation (23), and integration by parts to write

$$\int x\, p^*(x,t)\, \mu(x,t)\, dx = [(b-c)/b]\ CAL(t) \qquad (35)$$

Equations (20) and (33) then yield

$$A_d(t) \approx CAL(t) \qquad (36)$$

The approximate equality between $A_d(t)$ and CAL(t) parallels the exact equality found in the Bongaarts-Feeney mortality model [cf. equation (6)]. It also implies that $\Delta\, A_d(t) \approx c/b$, and that the population mean age at death increases linearly in the same manner as e_0 and CAL.

Generating Model Population Values Over Time. From the general expression in equation (11), we can write

$$p(x,t) = \exp[\ -e^{-ct}\ \{e^{A+bx} - e^{A+cx}\}\ /(b-c)] = p(x,0)\hat{}(e^{-ct}) \tag{37}$$

where the notation $z\hat{}y$ indicates z^y. At every age, the wedge-period population at time t is thus the wedge-period population at time 0 raised to the power e^{-ct}. As $c>0$ and $p(x,0)\leq 1$, equation (37) indicates that $p(x,t)\rightarrow 1$ as $t\rightarrow\infty$. That ultimate uniform population distribution implies zero mortality and is consistent with zero entropy.

The same procedure can be followed with respect to the number of survivors to age x in the life table based on the death rates prevailing at time t. Equation (9) can be rewritten

$$p_{LT}(x,t) = \exp[\ e^{-ct}\ \{(e^{A} - e^{A+bx})\ /\ b\}] = p_{LT}(x,0)\ \hat{}[e^{-ct}] \tag{38}$$

Every period life table can thus be generated directly from the time 0 life table.

The Special Case of b=c. Since b, the rate of increase in mortality over age, is approximately 0.1 in contemporary human populations, this special case is one of extremely rapid mortality decline. When b=c, the force of mortality is given by

$$\mu(x,t) = \exp[\ A + c(x-t)\ ] \tag{39}$$

Equation (39) implies that the death rate of the cohort born at time τ is $\exp[A-c\tau]$ at every age. The cohort probability of survivorship to age x at time t is

$$p(x,t) = \exp[\ -x\ e^{\{A+c(x-t)\}}\ ] = \exp[\ -x\ \mu(x,t)\ ] \tag{40}$$

Equations (9), (12), and (26) hold. Equation (14) is valid as well, with $\Delta e_0(t) \approx 1$.

The Bongaarts-Feeney Mortality Model With a Constant Rate of Decline. The present model is not a Bongaarts-Feeney mortality model. It expresses mortality decline in terms of the force of mortality function specified in equation (8), and all other functions follow from that specification. Neither the force of mortality function $\mu(x,t)$ nor the wedge-period population function $p(x,t)$ shifts along the x-axis over time. However, it is similar to a Bongaarts-Feeney model with linear shifts. If $\mu(x,0)=e^{A+bx}$ and the Bongaarts-Feeney shift function F(t) in equations (4) and (5) is (c/b)t, then the Bongaarts-Feeney force of mortality is given by $\mu_{BF}(x,t)=e^{A+bx-ct}$ (for $x \geq F(t)$) and

$$p_{BF}(x,t) = \exp[\ -e^{A-ct}(e^{bx} - e^{ct})/(b-c)\ ] \qquad , x \geq F(t) \qquad (41)$$

When $x<F(t)$, $\mu_{BF}(x,t)=0$ and $p_{BF}(x,t)=1$.

Equation (41) shows that the Bongaarts-Feeney model population at age x and time t [$x \geq (c/b)t$] differs from the present model population of equation (11) in terms of the last exponential term on the right (i.e. it has e^{ct} rather than e^{cx}). Even with Gompertz-type mortality and linear shifts, the sliding Bongaarts-Feeney model has a different form. The continually shifting age below which equation (41) does not apply means that equations (37) and (38), which generate model population sizes and period life table values, do not hold. Numerical differences may be modest, however. When ct and cx are considerably smaller than bx, they have little impact on population size and composition. Ignoring the e^{cx} term in equation (11) is equivalent to neglecting the D#(t) term in equation (15), or simplifying the population distribution using equation (21). That simplification takes advantage of the model's very low mortality at younger ages and, as we see below, generally produces good estimates.

Generalizing the Model to Allow Varying Rates of Mortality Decline. Because dynamic mortality models with a constant annual number of births do not involve feedbacks between mortality and fertility, equations (8)-(12) can be generalized to accommodate time varying changes in the rate of mortality decline. Given the force of mortality function

$$\mu(x,t) = \exp[\ A + bx - f(t)\ ] \qquad (42)$$

for a specified function f(t), the wedge-period populations can be found from the general survival relationship

$$p(x,t) = \exp[\, -\int_0^x \mu(a, t-x+a)\, da \,] \tag{43}$$

Relationships in such models are likely to be more complex than those developed here for the model of equation (8), but have yet to be analyzed in depth.

NUMERICAL ILLUSTRATIONS OF CONTINUOUSLY DECLINING MORTALITY

In equation (8), the force of mortality is defined in terms of two readily interpretable parameters: b, the rate of mortality increase over age and c, the rate of mortality decrease over time. Historically and contemporaneously, the value of b has been about 0.1. Recently mortality rates in a number of developed countries have declined by about 1% per year. Focusing on parameters with similar values, **Figure 1** shows force of mortality curves at different time points for 4 combinations of b and c. In all cases, the curves have a similar shape and the μ values decline over time. When b rises from 0.1 to 0.125, μ increases substantially (and the graph stops at age 110 instead of 120). When the rate of mortality decline increases from 0.008 to 0.012, μ values decrease appreciably for all t>0.

Figure 2 shows model population sizes and structures for the same 4 parameter combinations. The p(x,t) curves follow a similar overall pattern, though the level of survivorship varies substantially over time and between models. It is clear from the figures that while a Gompertz form force of mortality function leads to rapid rises in mortality over age, that form does not imply any limit to survivorship.

In **Figure 3**, we examine four different model functions, focusing on b=0.1 and the two previous values of c (that are a bit under and a bit over 1%). Panel A shows the nearly linear increase in CAL over time, and that the slopes of those curves are close to c/b. For example, when c=.008, the size of the population increases from 82.53 persons at time 0 to 98.52 persons at time 200, an average annual increase of .0799. The underlying values for those functions, and for three other model measures, are shown in **Table 1**.

Period life expectancy values are shown in Panel B of Figure 3. The slopes of the e_0 (t) curves are very similar to those of CAL(t), and their levels differ only slightly. With c=0.008, $e_0(0)$ is 83.36, larger than CAL(0) by 0.83. At time 200, e_0 (200) is 99.35, which again exceeds CAL(200) by 0.83. The steady increase in life expectancy, about one month per year when c=0.008, is not large compared with the long term trend found by Oeppen and Vaupel (2002).

Panel C shows values for the mean age at death, and virtually reproduces Panel A. Table 1 shows that, to two decimals, the values of A_d (t) are usually the same as those of CAL(t). With one birth per year, the model population's size is identical to its average age at death.

Panel D of Figure 3 shows values for the mean age of the population. The change over time is again close to linear, with a slope near c/2b. For example, with c=0.008, the mean population age increases from 42.25 at time 0 to 46.17 at time 100 and 50.09 at time 200. Between times 0 and 100, and between times 100 and 200, the average annual increase is 0.0392.

Table 1 shows the decline in the population's crude death rate, as declining mortality overrides the effects of population aging. With one birth per year, the population's crude birth rate also declines as population size increases. The population maintains a positive rate of natural increase each year, since the crude birth rate always exceeds the crude death rate. Although RNI(t) declines over time and is modest in size, it indicates that the population could continue growing even if fertility were slightly below replacement. The values in Table 1 confirm the approximate relationships in equations (28), (29), and (30), as entropy at time t is closely related to the crude rates of birth, death, and natural increase.

A concern often associated with mortality improvements is their impact on pension systems. Continuing mortality declines and the accompanying rise in the proportion of persons over 65 is virtually certain to lead to increases in the "normal" age of retirement. To illustrate how the present model can be used to analyze the population dynamics involved, we consider a simple scenario. Let W be the time 0 ratio of (i) the retired population (i.e. the number of persons above the initial retirement age of 65) to (ii) the economically active population (i.e. the number of persons between exact age 20 and retirement age). Then assume that every year the retirement age increases so that the retired population/active population ratio remains equal to W. **Figure 4** depicts how the age of retirement increases under that scenario for b=0.1 and four different values of c. The increase in the age of retirement is quite linear, with the slope approximately equal to 6.8c. With c=0.008, the retirement age rises from 65 to 70.4 years at time 100, and to 75.9 years at time 200. Under this scenario, years spent in the labor force and in retirement both increase by about the same factor. When c=0.008, the time 0 figures imply that a person age 20 can expect to live 44.1 out of the 45 active years, while a person age 65 can expect to live another 20.4 years retired. At time 100, the comparable figures are 49.7 years active and 22.5 years retired.

SUMMARY AND CONCLUSIONS

A Gompertz form dynamic population model can capture relationships in populations experiencing continually declining mortality. The cross-sectional population of the model can be termed a wedge-period population, because it is not a consequence of either period or cohort rates but of a "wedge" of rates. The form of change in many model variables is very nearly linear, and is driven by the ratio of mortality improvement parameter c to aging parameter b. As death rates retreat to ever higher ages, survivorship patterns change uniformly, retaining the same overall pattern. The population size and structure at any time, and the life table for any period, can be found from simple transformations of the time 0 values. Even with a constant annual number of births, the population steadily increases in size by an amount equal to c/b. At any time, its rate of natural increase is determined by its entropy value multiplied by parameter c. As the mean age of the population continues to rise, the age at retirement is likely to advance, and both can increase by a fixed annual amount that reflects the pace of mortality decline. As the stable population model has provided an analytical foundation for many demographic analyses, the present model of equation (8) can provide a simple, flexible framework for analyzing the implications of continually declining mortality on longevity and on the size and composition of the population.

ENDNOTES

1. There is a minor technical defect in the derivation of $\mu(x,t)$ given in Appendix 2 of Bongaarts and Feeney (2002). Since $\mu^*(x,t)=(\partial/\partial x) \ln p(x,t)$ is zero for $x \leq F(t)$, their equation (13b) should state

$$\mu(F(t),t) = (1 - dF(t)/dt)\, \mu^*(F(t),t) \tag{E.1}$$

2. Schoen and Kim (1992) provided a general method for finding the time derivatives of many demographic functions. Of particular relevance here, the derivative of the mean age of G (where G can denote population size or death) is given by

$$dA_G(t)/dt = A_G(t)\, [\, R_{Gx} - R_G \,] \tag{E.2}$$

where A_G is the mean age of G, R_{Gx} is the overall rate of increase of G times age, and R_G is the overall rate of increase of G. Equations (32) and (34) have the form of equation (E.2).

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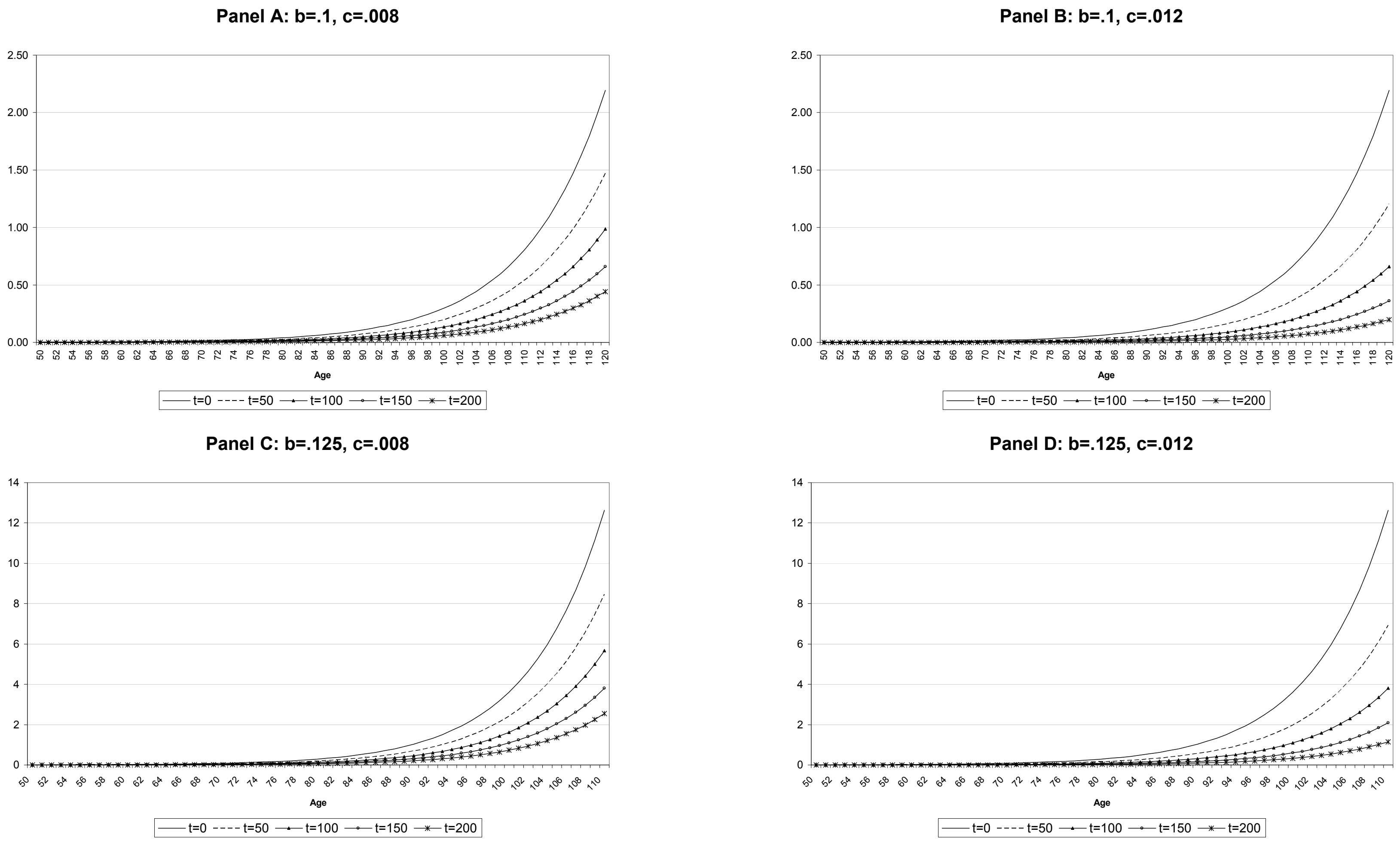
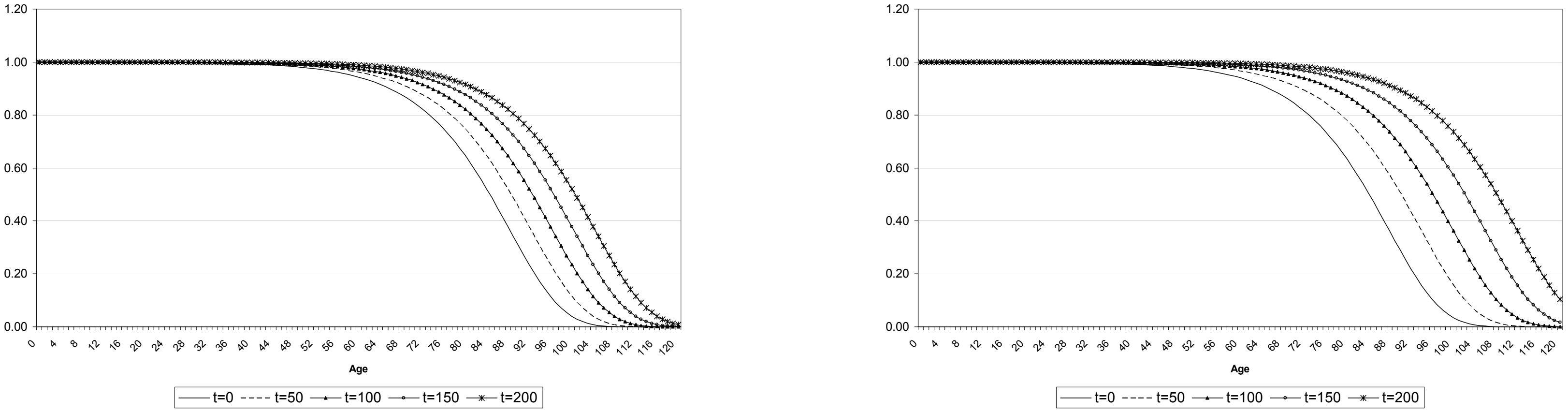
Figure 1. The force of mortality, μ(x,t) in model populations with different rates of mortality increase over age (x) and mortality improvement over time (t)

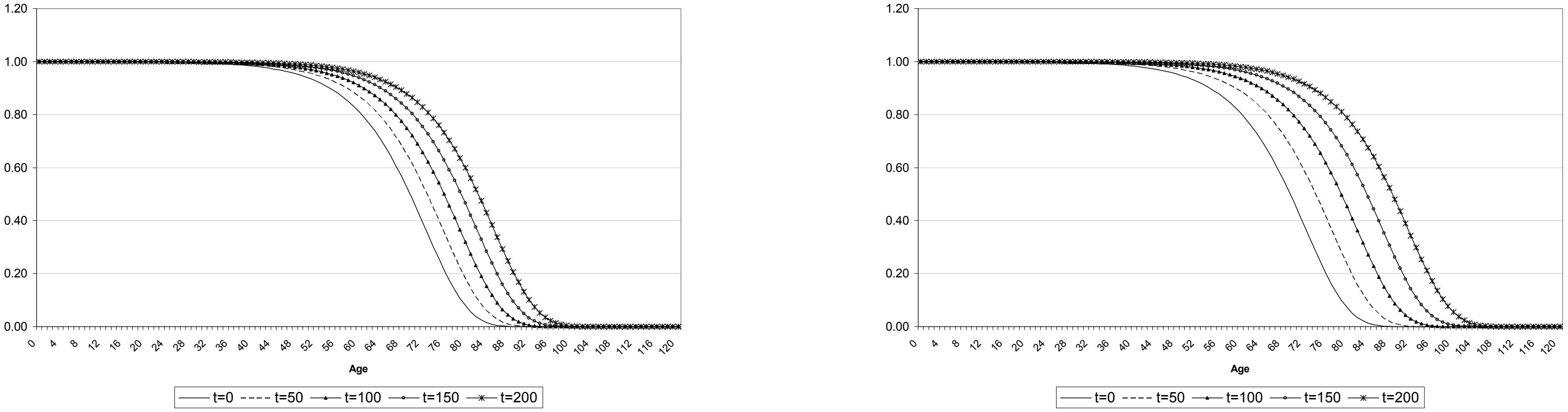


Note: The force of mortality is defined by $\mu(x,t)=e^{A+bx-ct}$. The value of A is fixed at -11.2146081, so that μ(50,0)=.002 when b=.1.

Figure 2. Model population size and structure [p(x,t)] with different rates of mortality increase over age (x) and mortality improvement over time (t)


Panel A: b=.1, c=.008

Panel B: b=.1, c=.012


Panel C: b=.125, c=.008

Panel D: b=.125, c=.012

Note: Population values generated using equation (11) with μ(x,t)=exp[-11.2146081+bx-ct].

Figure 3. Model population size, mean age, period life expectancy, and mean age at death, for b=.1 and for c=.008 and .012.

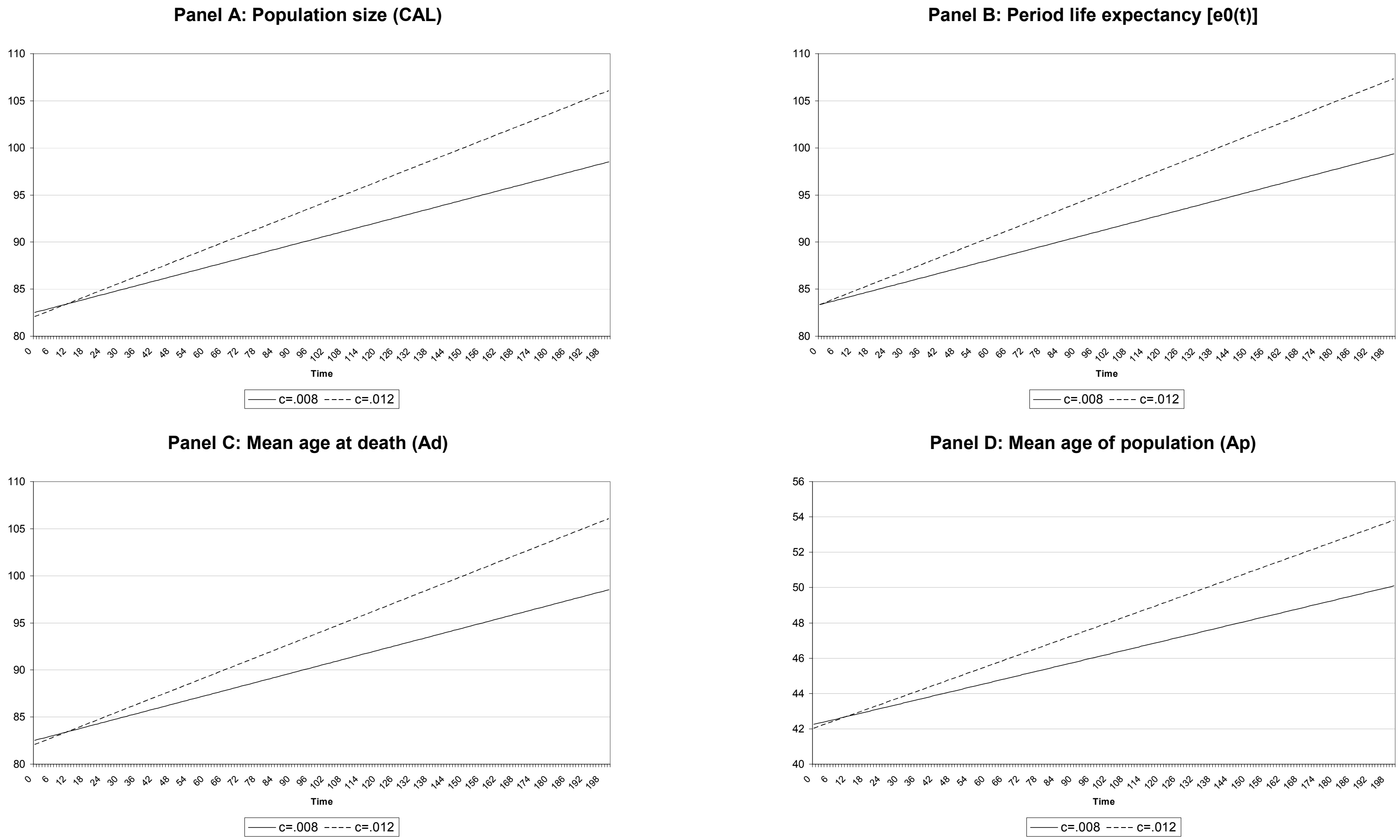


Figure 4. Retirement age needed to maintain a constant ratio of persons retired to persons in the labor force, model populations with b=.1 and different values of c

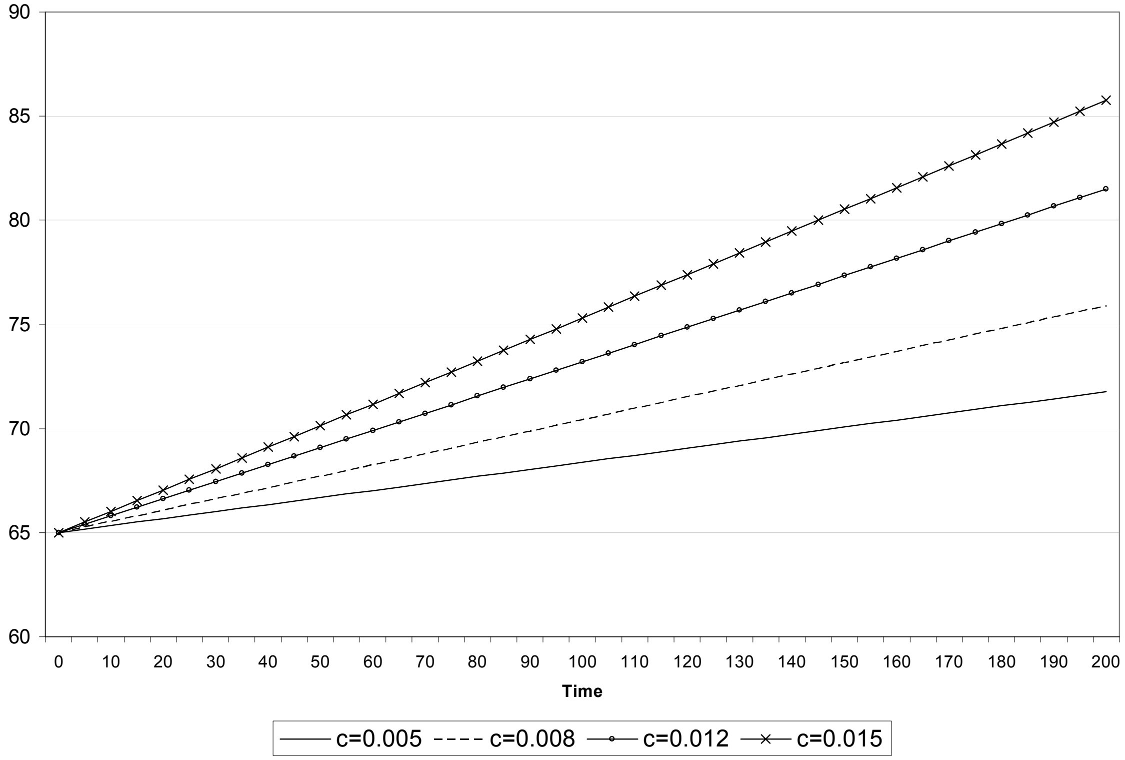


Table 1. Model population measures for b=0.1 and for c=.008 and .012

Panel A: c=.008

Time (t)	CAL(t)	$e_0(t)$	$A_d(t)$	$A_p(t)$	cdr(t)	rni(t)	Hp(t)
0	82.53	83.36	82.53	42.25	0.01115	0.00097	0.12097
5	82.93	83.76	82.93	42.45	0.01109	0.00096	0.12039
10	83.33	84.16	83.33	42.64	0.01104	0.00096	0.11982
15	83.73	84.56	83.73	42.84	0.01099	0.00095	0.11926
20	84.13	84.96	84.13	43.04	0.01094	0.00095	0.11870
25	84.53	85.36	84.53	43.23	0.01088	0.00095	0.11814
30	84.93	85.76	84.93	43.43	0.01083	0.00094	0.11759
35	85.33	86.16	85.33	43.62	0.01078	0.00094	0.11705
40	85.73	86.56	85.72	43.82	0.01073	0.00093	0.11651
45	86.13	86.96	86.12	44.01	0.01068	0.00093	0.11597
50	86.53	87.36	86.52	44.21	0.01063	0.00092	0.11544
55	86.93	87.76	86.92	44.40	0.01058	0.00092	0.11491
60	87.33	88.16	87.32	44.60	0.01054	0.00092	0.11439
65	87.73	88.56	87.72	44.79	0.01049	0.00091	0.11387
70	88.13	88.96	88.12	44.99	0.01044	0.00091	0.11336
75	88.53	89.36	88.52	45.19	0.01039	0.00090	0.11285
80	88.93	89.76	88.92	45.38	0.01035	0.00090	0.11235
85	89.32	90.15	89.32	45.58	0.01030	0.00089	0.11185
90	89.72	90.55	89.72	45.77	0.01025	0.00089	0.11136
95	90.12	90.95	90.12	45.97	0.01021	0.00089	0.11086
100	90.52	91.35	90.52	46.17	0.01016	0.00088	0.11038
105	90.92	91.75	90.92	46.36	0.01012	0.00088	0.10990
110	91.32	92.15	91.32	46.56	0.01007	0.00088	0.10942
115	91.72	92.55	91.72	46.75	0.01003	0.00087	0.10894
120	92.12	92.95	92.12	46.95	0.00999	0.00087	0.10847
125	92.52	93.35	92.52	47.15	0.00994	0.00086	0.10801
130	92.92	93.75	92.92	47.34	0.00990	0.00086	0.10755
135	93.32	94.15	93.32	47.54	0.00986	0.00086	0.10709
140	93.72	94.55	93.72	47.73	0.00982	0.00085	0.10663
145	94.12	94.95	94.12	47.93	0.00978	0.00085	0.10618
150	94.52	95.35	94.52	48.13	0.00973	0.00085	0.10574
155	94.92	95.75	94.92	48.32	0.00969	0.00084	0.10529
160	95.32	96.15	95.32	48.52	0.00965	0.00084	0.10485
165	95.72	96.55	95.72	48.72	0.00961	0.00084	0.10442
170	96.12	96.95	96.12	48.91	0.00957	0.00083	0.10398
175	96.52	97.35	96.52	49.11	0.00953	0.00083	0.10355
180	96.92	97.75	96.92	49.31	0.00949	0.00083	0.10313
185	97.32	98.15	97.32	49.50	0.00945	0.00082	0.10271
190	97.72	98.55	97.72	49.70	0.00942	0.00082	0.10229
195	98.12	98.95	98.12	49.90	0.00938	0.00081	0.10187
200	98.52	99.35	98.52	50.09	0.00934	0.00081	0.10146

Note: Symbols refer to functions defined in eqns (12), (10), (33), (31) (18), (18), and (27)

Table 1 (con't). Model population measures for b=0.1 and for c=.008 and .012

Panel B: c=.012

Time (t)	CAL(t)	$e_0(t)$	$A_d(t)$	$A_p(t)$	cdr(t)	rni(t)	Hp(t)
0	82.09	83.36	82.08	42.04	0.01072	0.00146	0.12158
5	82.69	83.96	82.68	42.33	0.01064	0.00145	0.12071
10	83.29	84.56	83.28	42.62	0.01057	0.00144	0.11985
15	83.89	85.16	83.88	42.92	0.01049	0.00143	0.11901
20	84.49	85.76	84.48	43.21	0.01042	0.00142	0.11817
25	85.09	86.36	85.08	43.50	0.01034	0.00141	0.11735
30	85.69	86.96	85.68	43.80	0.01027	0.00140	0.11654
35	86.29	87.56	86.28	44.09	0.01020	0.00139	0.11574
40	86.89	88.16	86.88	44.38	0.01013	0.00138	0.11494
45	87.48	88.76	87.48	44.68	0.01006	0.00137	0.11416
50	88.08	89.36	88.08	44.97	0.00999	0.00136	0.11339
55	88.68	89.96	88.68	45.26	0.00992	0.00135	0.11263
60	89.28	90.55	89.28	45.56	0.00986	0.00134	0.11188
65	89.88	91.15	89.88	45.85	0.00979	0.00133	0.11114
70	90.48	91.75	90.48	46.14	0.00973	0.00132	0.11041
75	91.08	92.35	91.08	46.44	0.00966	0.00132	0.10969
80	91.68	92.95	91.68	46.73	0.00960	0.00131	0.10898
85	92.28	93.55	92.28	47.03	0.00954	0.00130	0.10828
90	92.88	94.15	92.88	47.32	0.00948	0.00129	0.10758
95	93.48	94.75	93.48	47.62	0.00941	0.00128	0.10690
100	94.08	95.35	94.07	47.91	0.00935	0.00127	0.10622
105	94.68	95.95	94.67	48.20	0.00930	0.00127	0.10555
110	95.28	96.55	95.27	48.50	0.00924	0.00126	0.10489
115	95.88	97.15	95.87	48.79	0.00918	0.00125	0.10424
120	96.48	97.75	96.47	49.09	0.00912	0.00124	0.10359
125	97.08	98.35	97.07	49.38	0.00907	0.00124	0.10295
130	97.68	98.95	97.67	49.68	0.00901	0.00123	0.10232
135	98.28	99.55	98.27	49.97	0.00896	0.00122	0.10170
140	98.88	100.15	98.87	50.27	0.00890	0.00121	0.10109
145	99.48	100.75	99.47	50.56	0.00885	0.00121	0.10048
150	100.07	101.35	100.07	50.86	0.00879	0.00120	0.09988
155	100.67	101.95	100.67	51.15	0.00874	0.00119	0.09929
160	101.27	102.55	101.27	51.45	0.00869	0.00118	0.09870
165	101.87	103.15	101.87	51.74	0.00864	0.00118	0.09812
170	102.47	103.75	102.47	52.04	0.00859	0.00117	0.09755
175	103.07	104.35	103.07	52.33	0.00854	0.00116	0.09698
180	103.67	104.95	103.67	52.63	0.00849	0.00116	0.09643
185	104.27	105.55	104.27	52.92	0.00844	0.00115	0.09587
190	104.87	106.15	104.87	53.22	0.00839	0.00114	0.09533
195	105.47	106.75	105.47	53.51	0.00834	0.00114	0.09478
200	106.07	107.35	106.07	53.81	0.00830	0.00113	0.09425

Note: Symbols refer to functions defined in eqns (12), (10), (33), (31) (18), (18), and (27)