Using the Lee-Carter Method to Explain Linear Increases in Life Expectancy

Shripad Tuljapurkar, Stanford University
Nan Li, Max Planck Institute for Demographic Research

Recently, life expectancy rising approximately linearly has been observed in many low-mortality nations, we find that this phenomena is a conditional output of k(t) changing linearly in the Lee-Carter method, which characterizes stable mortality declines across all ages. The conditions for such an output to occur are low mortality at young ages, and at old ages the Gompertz law works and mortality decline at rates that are constant across age and over time. Using data of the G7 countries, we illustrate that these conditions roughly stand since 1950s, and show that the changes of k(t)s are as linear as that of life expectancy. In general, we do not recommend using life expectancy as the predictor to forecast mortality, because it does not provide age-specific death rates, it could hardly be more linear than k(t), and to be as linear as k(t) it requires three conditions that may not stand widely.

Introduction

Recently, Lee and Carter (1992) developed a (LC) method to forecast age-specific death rates (hereafter abbreviated as death rates) stochastically. In the LC method, death rates at ages *x* and time *t*, *m(x,t)*, are modeled as:

$$\log[m(x,t)] = a(x) + b(x)k(t) + \varepsilon(x,t), \qquad (1)$$

which can be estimated using the singular-value decomposition (SVD). This approach transforms the task of forecasting an age-specific vector *log[m(x,t)]* into forecasting a scalar *k(t)*, with small errors $\varepsilon(x,t)$. In other words, *k(t)* is the latent variable that the LC method uses to describe historical changes in death rates, and to forecast mortality.

According to (1), linear-over-time declines in *log[m(x,t)]*, with different slopes at each age *x*, lead to linear decline in *k(t)*, while the different age-specific slopes are described by *b(x)*. Thus, the condition for *k(t)* to decrease linearly is that *log[m(x,t)]* decline linearly, or that individual death rates decline at constant exponential rates.

In reality, death rates do not decline at constant rates, but fluctuate randomly around some average trends that may or may not be linear. These trends are transformed by the SVD model into an average trend of *k(t)*. If the average trends in of *log[m(x,t)]* are linear, then the average trend in *k(t)* will also be linear. The age-correlated random fluctuations of individual death rates around their trends are transformed into random fluctuations in *k(t)* around its trend. We will say that mortality decline is stable when the rates of decline of individual death rate can be represented in this way as constants over time plus random fluctuations. Then, the condition for *k(t)* to have an average linear trend is that death rates decrease stably. In fact, a linear trend in *k(t)* has been estimated for the U.S. (Lee and Carter, 1992) and for other G7 countries (Tuljapurkar, Li and Boe, 2000), which makes the LC method attractive for forecasting.

Using data from the U.S. between 1900 to 1989 and simulation analysis, Lee and Carter showed that the linear decline in *k(t)* over the 20th century led life expectancy at birth, $e_0(t)$, to increase faster at earlier times and slower later, or convexly. They suggested that such a relationship between *k(t)* and $e_0(t)$ may be understood in terms of mortality 'entropy' (Keyfitz, 1977: 62—68). As death rates at younger ages decline more rapidly to low levels, the lives saved by further stable mortality decline are mainly at older ages where they generate smaller increases in e_0.

Two related articles on the trends of $e_0(t)$ were published in 2002. One was written by Oeppen and Vaupel (2002), which shows that the highest values of $e_0(t)$ among human population, taken by different country in different period, compose a remarkable linear trend from 1840 to 2000. Another article, by White (2002), investigates linear trends in various mortality measures for 21 industrial nations from 1955 to 1996. White shows that linear trend fits $e_0(t)$ better than does the age-standardized death rate or the *log[m(x,t)]*.

The longest linear trend, among the above papers related with the trends of $e_0(t)$ and *k(t)*, is found in life expectancy by Oeppen and Vaupel, about 160 years. This trend, however, is not observed from a certain country in the whole period, but from different countries with the highest $e_0(t)$ in various sub-periods. Successively, these countries are Norway, New Zealand, Norway, Iceland, Switzerland and Japan. Along this linear trend, a certain nation replaces a leading country and then be replaced by another only when its $e_0(t)$ rises first faster and then slower, than this linear trend. In other words, leading countries' $e_0(t)$ should increase convexly, as also shown clearly by Oeppen and Vaupel using cases of Norway, New Zealand, the U.S. and Japan. Thus, Oeppen and Vaupel' finding is not inconsistent with Lee and Carter's analysis. The second longest linear trend is in *k(t)*, not $e_0(t)$, about 90 years, observed by Lee and Carter in the U. S. data. In the paper of Tuljapurkar, Li and Boe, linear trends were also found in *k(t)* in the other G7 countries in years between 1950 to 1994. White's paper is the first one that reports linear trends in certain populations' $e_0(t)$, using data from periods and countries similar with that used by Tuljapurkar, Li and Boe.

In White's paper, mortality indexes other than $e_0(t)$, such as age-standardized death rate and *log[m(x,t)]*, are discussed. The age-standardized death rate involves age structure that usually has nothing to do with mortality at the time of interest, and hence may not be relevant for discussing mortality trends. The exception is using the age structure of a stationary population, which is subjected to the mortality of interest and close to migration. Using this stationary age structure, the age-standardized death rate is $1/e_0(t)$. Thus, linear change in $e_0(t)$ corresponds to nonlinear change in the age-standardized death rate, and vice versa. Deviations from this relationship are results from factors other than mortality change.

In comparing the linear trends between $e_0(t)$ and *log[m(x,t)]*, White first regress them to linear lines, and then use the R-square value of regression to evaluate how linearly that $e_0(t)$ or *log[m(x,t)]* changes over time. Since the R-square value is the fraction of the variable's variance that is explained by the regression model, it tends to over estimate the linearity of $e_0(t)$ than *log[m(x,t)]*. To illustrate this, we may consider two functions. Let the first one be a linear function plus large random fluctuations, and the second one the same linear function plus a small quadratic term. It is obvious that the average trend of the first function is linear and of the second is nonlinear; it is equally obvious that the R-square value of first one is smaller, because its random component is larger. Turning back to *log[m(x,t)]* and $e_0(t)$, the formers include random fluctuations in mortality change, which tend to cancel each other in the process of calculating the later. This would lead to smaller R-square values of *log[m(x,t)]* than $e_0(t)$, even if the average trends of *log[m(x,t)]* were closer to linear than that of $e_0(t)$. Despite these details, however, White showed clear linear trends in the changes of $e_0(t)$ in 21 industrialized countries from 1955 to 1996.

On the other hand, Tuljapurkar, Li and Boe displayed equally clear linear trends in *k(t)* for the G7 countries in the period of 1950 to 1994. Different from the situation of $e_0(t)$, the condition for *k(t)* to show linear trend is known as that mortality declines stably. And why linear trends in *k(t)* were observed in the G7 countries in 1950 to 1994 seems understandable: these countries reduced their mortality stably in the period of interest.

The questions are, can $e_0(t)$ and *k(t)* be both close to linear, and which one is closer to linear? We answer these questions using data from the G7 countries.

Comparing the linearity between $e_0(t)$ and *k(t)*

We use mortality data from the G7 countries between 1950 and 1994, the period similar to that in White's paper, to compare the linearity between $e_0(t)$ and *k(t)*. Instead of using linear regression and its R-square value, we do the comparison in a more direct way. We first standardize $e_0(t)$ and *k(t)* as *Eo(t)* and *K(t)*, respectively:

$$Eo(t) = \frac{e_0(t) - e_0(1950)}{e_0(1994) - e_0(1950)}, \quad K(t) = \frac{k(t) - k(1950)}{k(1994) - k(1950)}. \qquad (2)$$

In other words, the linearity of *Eo(t)* or *K(t)* is identical to $e_0(t)$ or *k(t)*, respectively; but both *Eo(t)* and *K(t)* rise from 0 to 1 in the interested period in order to be comparable.

We then plot the values of *Eo(t)* and *K(t)* in Figures 1 to 7, for the G7 countries respectively. Expect Italy and Japan, the values of *Eo(t)* and *K(t)* are almost identical for all *t*, regardless of how closer to linear that they are. It can also be seen that *K(t)* is closer to linear than *Eo(t)* for Japan without arguing how to quantitatively measure linearity, but not so for Italy. The questions about can the $e_0(t)$ and *k(t)* be both close and which one is closer linear are answered: that the $e_0(t)$ and *k(t)* are similarly linear in the G7 countries in the period of 1950 to 1994.

This answer, however, raises other two questions. First, as the changes in *k(t)* is linearly related with the decline rates of death rates, which determine the change of $e_0(t)$ in the way that is so nonlinearly, how can *k(t)* and $e_0(t)$ be so similar topologically? In other words, how can the *Eo(t)* and *K(t)* be so close? Second, how to explain Lee and Carter's observation that *k(t)* is closer to linear than $e_0(t)$? We answer the two questions using the LC method.

Explanation

An exact relationship

We need two conditions to build an exact relationship between the changes of life expectancy and the force of mortality. The first condition is the Gompertz formula that mortality force at age *x* and time *t*, $\mu(x,t)$, rises with age exponentially from some starting age *A*, say 40,

$$\mu(x,t) = \mu(A,t)\exp(\theta(x-A)). \tag{3}$$

The second condition is that at time *t* and for all ages *x>A*, mortality forces decline by the same rate *D(t)*,

$$D(t) = -\frac{1}{\mu(x,t)}\frac{d}{dt}\mu(x,t) = -\frac{1}{\mu(A,t)}\frac{d}{dt}\mu(A,t). \tag{4}$$

Given (3) and (4), the change of life expectancy at age *A*, $e_A(t)$, is described in the Appendix as

$$\frac{d}{dt}e_A(t) = \frac{D(t)}{\theta}[1-\mu(A,t)e_A(t)] \tag{5}$$

An accurate relationship

For values of *A* around 40, $\mu(A,t)e_A(t)$ is small, the discrete version of a non-exact, but accurate relationship between changes of life expectancy and death rates is obtained from (5) as

$$\Delta e_A(t) = e_A(t+1) - e_A(t) \approx \frac{D(t)}{\theta}. \qquad (6)$$

Equation (6) cannot be accurate for life expectancy at birth without additional condition, since infant mortality is usually higher than that at young-adult ages and hence Gompertz formula does not well apply at very young ages.

The third condition, for (6) to be accurate for life expectancy at birth, is that death rates at ages younger than *A* are small and negligible in determining $e_0(t)$. Under this condition, life expectancy at birth and time *t* is approximately $A + e_A(t)$. Thus, according to (6)

$$\Delta e_0(t) = e_0(t+1) - e_0(t) \approx [A + e_A(t+1)] - [A + e_A(t)] = e_A(t+1) - e_A(t) \approx \frac{D(t)}{\theta}. \qquad (7)$$

Vaupel (1986) obtained (7) in a different way.

In summary, there are three conditions for (7) to be accurate. The first one is that Gompertz formula applies to ages older than *A*. The second is that the death rates at ages older than *A* decline by the same rate. The third is that death rates at ages younger than *A* are small and negligible in determining $e_0(t)$.

An explanation

According to (1), the values of *b(x)* and *k(t)* can be rescaled as long as their product maintains constant. We may let *b(x)=1* for ages *x>A*. Thus, death rates declining by the same rate *D(t)* at ages *x>A* is described by that *k(t)* declines *D(t)*. This decline in *k(t)*, together with the other two conditions that Gompertz formula applies at ages older than *A* and there are no mortality at ages younger than *A*, lead to $e_0(t)$ to increase $D(t)/\theta$, according to (7). In the standardization, the increase of *K(t)* in one year is $(-D(t))/\sum(-D(s))$ =$D(t)/\sum D(s)$, and of *Eo(t)* is $[D(t)/\theta]/\sum[D(s)/\theta]$ =$D(t)/\sum D(s)$, which is the same as of *K(t)*. Since *K(t)* and *Eo(t)* start with the same value zero, they should be close to identical over the period of interest.

This may explain the closeness between *Eo(t)* and *K(t)* in the G7 countries other than Italy and Japan. The small differences between *Eo(t)* and *K(t)* in these countries should due to that *b(x)* in the LC model are not constant over age for ages, say, older than 40. It could also due to that the Gompertz law does not perfectly apply, that $\mu(A,t)e_A(t)$ is not zero, and that there are mortality at ages younger than *A*, especially at very young ages. In addition, mortality changes corresponding to higher-order SVD terms require adjusting *k(t)* to fit the recorded $e_0(t)$, which may make further difference between *Eo(t)* and *K(t)*.

When the third condition that death rates are small at younger ages is far away from true, the difference between *Eo(t)* and *K(t)* would be larger, and *Eo(t)* would be larger than *K(t)* because more mortality decline at younger ages would lead to larger increase in $e_0(t)$. This may explain the cases of Italy and Japan, as their *b(x)* at older ages are similar with the other G7 countries in Figure 8, but their death rates at younger ages are remarkably higher, as can be seen in Table 1.

Table 1. Values of of death rates at ages 0 and 1—4 of the G7 countries, year 1950

	Canada	France	Germany	Italy	Japan	UK	US
m(0)	.043	.051	.050	.070	.058	.031	.033
m(1-4)	.002	.002	.002	.005	.010	.001	.001

The explanations may be summarized as following:

When death rates at younger ages are small, the *Eo(t)* and *K(t)* would be closer, or the $e_0(t)$ and *k(t)* would be topologically similar, whether or not linear. This may explain situations of most developed countries at times later than 1950s.

When death rates at younger ages are higher, *Eo(t)* and *K(t)* would no longer be close. Linear decline in *k(t)* would lead to $e_0(t)$ to increase convexly, because the additional mortality decline at younger ages. This may explain the results in the paper of Lee and Carter, and in Japan here. However, faster than linear decline in *k(t)* may result in $e_0(t)$ to increase in the way that is closer to linear than *k(t)*. Italy is of this case, where the *Eo(t)* is closer to linear, but the *k(t)* declines faster than linear, because that the *K(t)* is lower than linear is resulted from dividing *k(t)* by a negative number. This closer to linear increase in $e_0(t)$ is because that death rates at older ages declined faster at later times, which maintained the faster rise pattern of $e_0(t)$ at earlier times. Further, slower than linear decline in *k(t)* may result in $e_0(t)$ that is less linear than *k(t)*. Japan provides an example for this case if we pay attention to the small difference between *K(t)* and the linear line.

Discussion

The direct measure of mortality is age-specific death rate. That age-specific death rates decrease over time at rates around constant values, as results of populations stably reduce mortality, are characterized by the linear decline of *k(t)* in the LC method. Such linear trend in *k(t)* is observed in the U.S. for 90 years, and in other G7 countries for more than 40 years.

Whether or not the linear decline in *k(t)* results in linear increase in $e_0(t)$ depends on three conditions. The first one is that the Gompertz formula applies to ages older *A*, say 40. The second is that the death rates at ages older than *A* decline by the same rate; and the third is that death rates at ages younger than *A* are small and negligible in determining

$e_0(t)$. If the three conditions stood, not only that linear declines in *k(t)* yield linear increase in $e_0(t)$, but that the changes in *k(t)* and $e_0(t)$ are topologically identical, regardless of linear or not.

Among the three conditions, the Gompertz formula is perhaps always approximately true for ages older than 40. Demographers have been using the Gompertz formula (see Preston, Heuveline and Guillot, 2001: 192) for almost two hundred years; recent modifications on this formula are only on death rates at the oldest ages, usually 90 years and older (e.g., Horiuchi and Coale, 1990), which may have only minor effects on life expectancy at birth.

The second condition, that death rates at ages older than 40 decline at approximately the same rate, or that the *b(x)* in (1) maintain constant over *x* for *x>40*, are also empirically true. The values of *b(x)* are roughly constant for *x>40* for the G7 countries in the period of 1950 to 1994, as can be seen in Figure 8. For longer period, time-varying *b(x)* may describe death rates better, but the values of *b(x)* for *x>40* are still approximately constant over *x* (Lee and Miller, 2001).

It is well known that mortality decline at younger ages would move populations into a new demographic era, in which increases of life expectancy would mainly depend on declines of death rates at older ages. Thus, the third condition, that death rates at ages younger than 40 are small and negligible in determining $e_0(t)$, would be gradually approached by developed countries first, and developing countries later. This transition will lead to a new phenomena, that is the changes in $e_0(t)$ and *log[m(x,t)]* would be topologically similar, whether or not linear. The third condition seems to be approximately true for most of the G7 countries since 1950s, thus changes in $e_0(t)$ and *log[m(x,t)]* are topologically similar in these countries. Of course, the linear declines in *log[m(x,t)]* observed by Tuljapurkar, Li and Boe should lead to the linear increases in $e_0(t)$ reported by White.

The third condition, however, has been less approximately true for Italy and Japan, as shown in Table 1. Therefore, the changes in *K(t)* and *Eo(t)* are less similar in Italy and Japan than in other G7 countries.

The third condition, obviously, cannot be approximately true for developed countries in earlier 20th century, when death rates at young ages were high and declining notably. In this situation, the increase in $e_0(t)$ additional to that of (7), caused by notable declines in death rates at young ages, would occur. For linear decline in *k(t)* and larger values of *b(x)* that drive faster mortality decline at younger ages, the additional increase in $e_0(t)$ would be larger at earlier times when death rates at younger ages were higher, and smaller at later times after the reduction of death rates at younger ages. This explains the situation in which linear *k(t)* yields convex increase in $e_0(t)$, as observed by Lee and Carter using the U.S. data. The entropy analysis, as suggested by Lee and Carter, could also explain the convex increase of $e_0(t)$, through numerical calculations of life table entropy for particular cases.

The third condition cannot be approximately true for developing countries in recent times either, where death rates at young ages are high and declining notably. In developing countries, stably reducing mortality can still be expected, and hence linear trends in *k(t)* may also well describe mortality change. Such linear trends in *k(t)*, however, would lead $e_0(t)$ to increase convexly, not linearly.

Comparing linearity between $e_0(t)$ and other mortality indexes may lead to suggestions about which one should be chosen as predictor, and result in different mortality forecasts, which may matter in socioeconomic business, not only mathematical properties. It is a common sense that simpler changing predictor should be more likely to produce better forecast. According to this common sense, closer to linear changing index should be chosen as the predictor to forecast mortality. We find, however, that $e_0(t)$ were neither closer to linear than *k(t)* in earlier times, nor so recently in the G7 countries.

There is a rare case in which $e_0(t)$ can be closer to linear than *k(t)*: when *k(t)* declines faster than linearly and death rates at younger ages are high. Italy may be an example for this situation, though its $e_0(t)$ is just slightly closer to linear than *k(t)*. Assuming that $e_0(t)$ could be a predictor better than *k(t)* for this situation, we can clarify the basic difference between choosing $e_0(t)$ and *k(t)* as predictor. In our view, the better basis for the LC method to forecast exists when death rates change more stably and the average trend in *k(t)* is more linear. On a better basis, using *k(t)* as predictor would be better than using $e_0(t)$, because *k(t)* is closer to linear than $e_0(t)$ in this case. When death rates change less stably and the average trend in *k(t)* is less linear, the basis for the LC method to forecast is worsen. On a worse basis, the LC method picks out linear component maximally from the average trend, and treats the nonlinear component in the average trend as random change. In other words, the LC method would produce a worse forecast on a worse basis, though how to handle non-linear average trends in *k(t)* is an issue of improving the LC method. On a worse basis for the LC method to forecast, $e_0(t)$ could be closer to linear than *k(t)*, and hence using $e_0(t)$ could make better forecast than using *k(t)*, as assumed. Thus, using $e_0(t)$ could be better than using *k(t)* when death rates change less stably, and would be worse than using *k(t)* when death rates change more stably. How can a predictor work better when the basis of the LC method is worse, and work worse when the basis of the LC is better? The answer is that this predictor does not stand on the basis that death rates change stably, but on something else.

The similarity, between the changes of logged death rates and life expectancy, in general does not suggest using life expectancy as predictor to forecast mortality. If the similarity were high, such as in most developed countries recently, using $e_0(t)$ to replace *k(t)* as predictor would be unnecessary, since it may make only negligible difference. When the similarity were low and mortality declines stably, *k(t)* would be closer to linear than $e_0(t)$, using $e_0(t)$ as predictor would not be an issue.

When the similarity were low and *k(t)* declines faster than linearly, $e_0(t)$ could be closer to linear than *k(t)*. In this case, using $e_0(t)$ as predictor would have to assume that death rates decline at accelerate rates, not constant rates. Although it seems to be consistent with the common sense that simpler trend would lead to better forecast at the level of $e_0(t)$; it is inconsistent with the same common sense at the most fundamental level, because the changes in age-specific death rates are assumed more complicated than that at constant rates.

In summary, mortality decline at younger ages would move populations into a new demographic era, in which rises of life expectancy would depend mainly on declines of mortality at older ages. This era may begin in 1950s in the G7 countries, and will start in developing countries later. In this era, declines of death rates at constant rates would no longer cause life expectancy to increase convexly, but linearly. Further, the changes of logged death rates and life expectancy would be topologically similar, whether or not linear. What could be developed from using the similarity between the changes of logged death rates and life expectancy remains to be explored; however, this similarity does not suggest using the later as the predictor of mortality.

Appendix

At time *t* and for ages older than *A*, suppose that the age-specific mortality force $\mu(x,t)$ changes with age *x* according to the Gompertz formula,

$$\mu(x,t) = \mu(A,t)\exp[\theta(a-A)], \quad x > A, \tag{1a}$$

whose over-time decline rates are constant across age,

$$D(t) = -\frac{1}{\mu(x,t)}\frac{d}{dt}\mu(x,t) = -\frac{1}{\mu(A,t)}\frac{d}{dt}\mu(A,t). \tag{2a}$$

The survivorship function *l(x,t)* and the life expectancy at age *A*, $e_A(t)$, are defined as usual by

$$\begin{aligned} l(x,t) &= l(A,t)\exp[-\int_A^x \mu(y,t)dy] \\ &= l(A,t)\exp\{-\frac{\mu(A,t)}{\theta}[\exp(\theta(x-A))-1]\}, \quad x > A \end{aligned} \tag{3a}$$

$$e_A(t) = \frac{\int_A^\infty l(x,t)dx}{l(A,t)} = \frac{\exp[\frac{\mu(A,t)}{\theta}]}{\theta} \int_{\frac{\mu(A,t)}{\theta}}^\infty \frac{\exp(-x)}{x} dx . \quad (4a)$$

According to (4a), there is

$$\frac{d}{dt} e_A(t) = [\frac{e_A(t)}{\theta} - \frac{1}{\theta\mu(A,t)}] \frac{d}{dt} \mu(A,t) . \quad (5a)$$

Using (2a), (5a) is written as

$$\frac{d}{dt} e_A(t) = \frac{D(t)}{\theta} [1 - \mu(A,t) e_A(t)] . \quad (6a)$$

References

Oeppen, J. and J. W. Vaupel. 2002. Broken limits to life expectancy. Science 296: 1029—31.

White, K. M. 2002. Longevity advances in high-income countries, 1955—96. Population and development Review 28(1): 59-76.

Lee, R. D. and L. Carter, 1992. Modeling and Forecasting the Time Series of U.S. Mortality. Journal of the American Statistical Association 87: 659—71.

Lee, R. D. and T. Miller, 2001. Evaluating the Performance of the Lee-Carter method for Forecasting Mortality. Demography 38: 537—49.

Tuljapurkar, S., N. Li and C. Boe, 2000. A Universal Pattern of Mortality change in the G7 Countries. Nature 405:789—92.

Vaupel, J. W. 1986. How change in age-specific mortality affects life expectancy. Population Studies 40: 147—57.

Keyfitz, N. 1977. *Applied mathematical demography*. Now York: John Wiley.

Figure 1. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for Canada

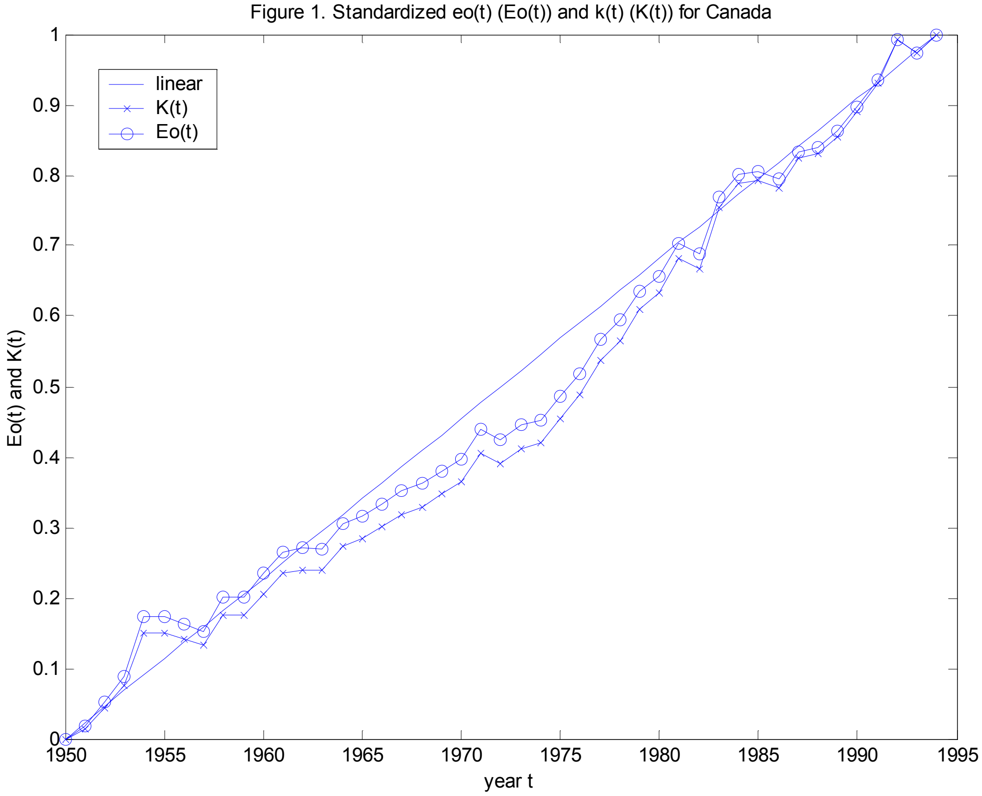
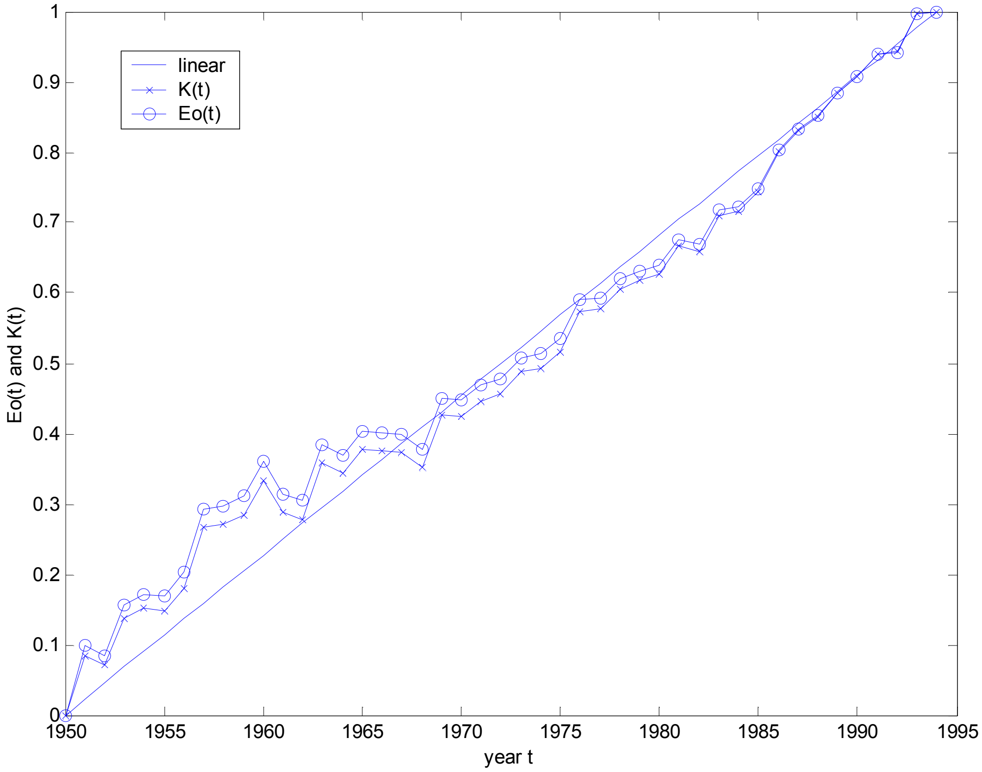


Figure 2. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for France



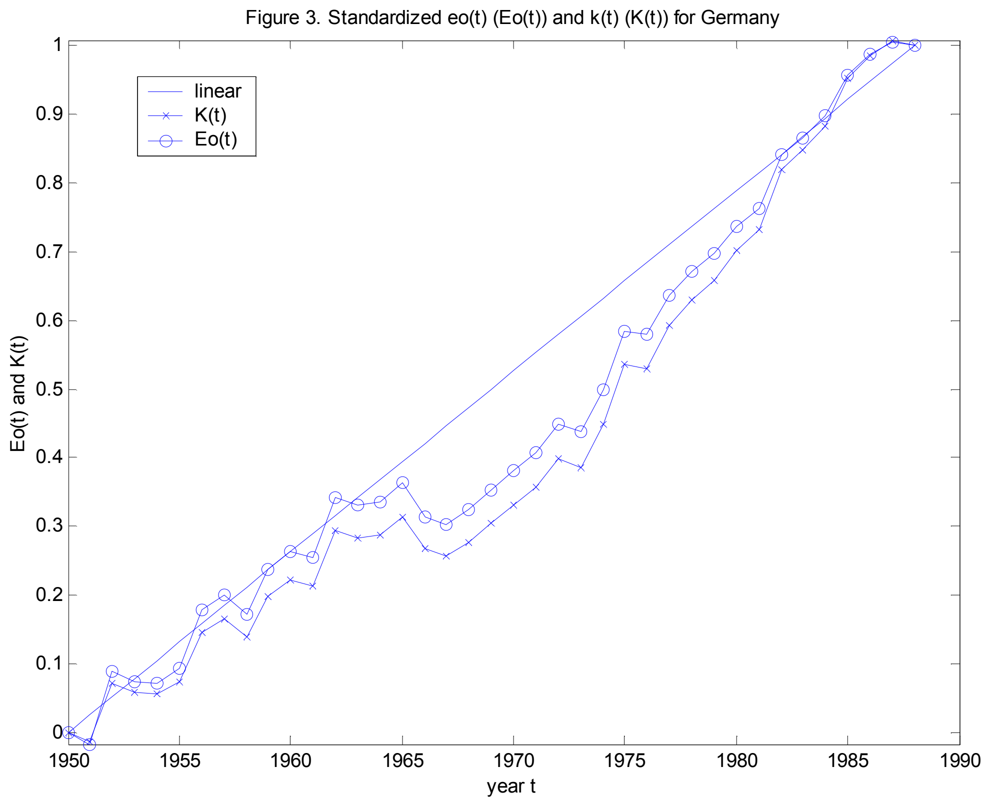


Figure 3. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for Germany
linear
K(t)
Eo(t)
Eo(t) and K(t)
year t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1950
1955
1960
1965
1970
1975
1980
1985
1990

Figure 4. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for Italy

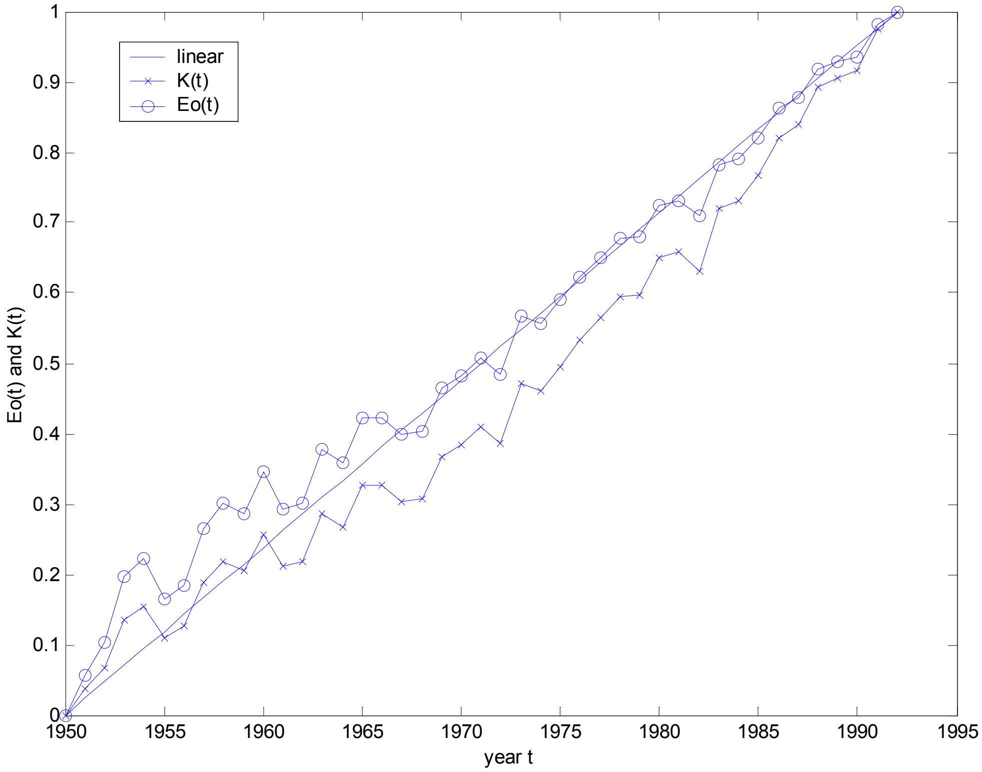


Figure 5. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for Japan

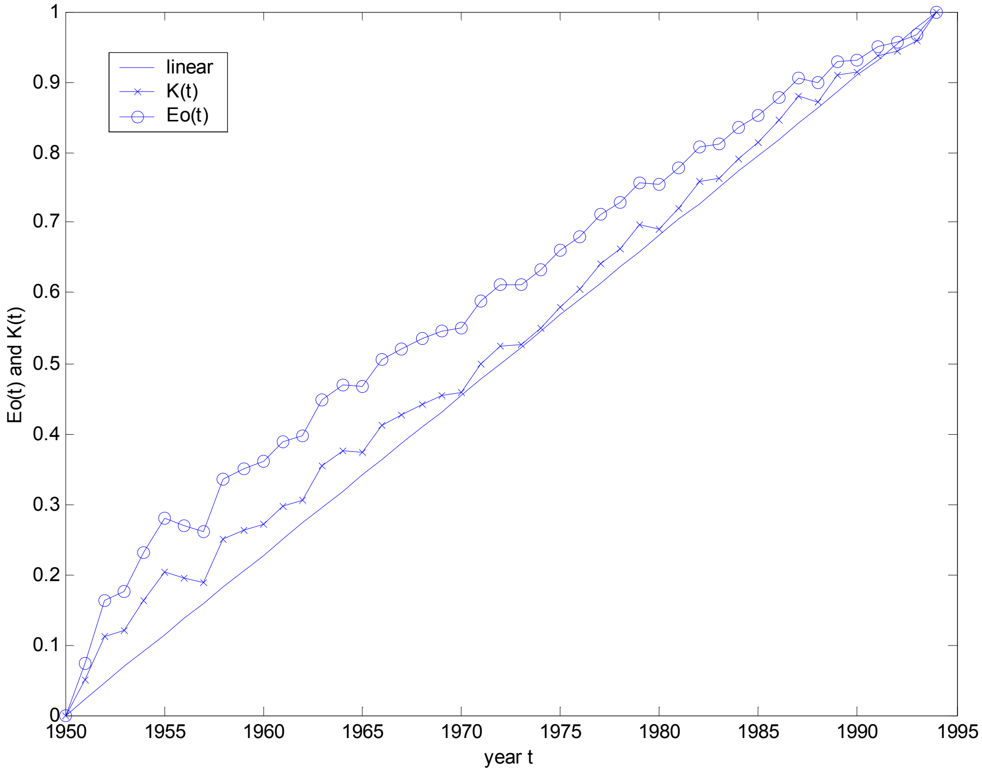


Figure 6. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for UK

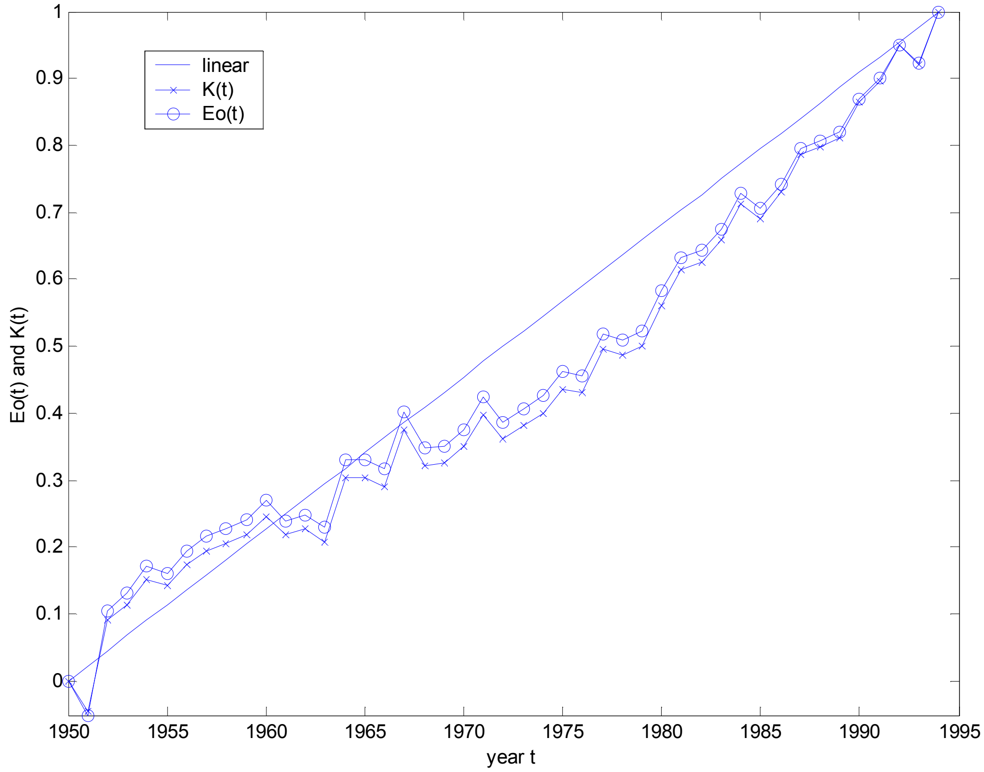


Figure 7. Standardized eo(t) (Eo(t)) and k(t) (K(t)) for US

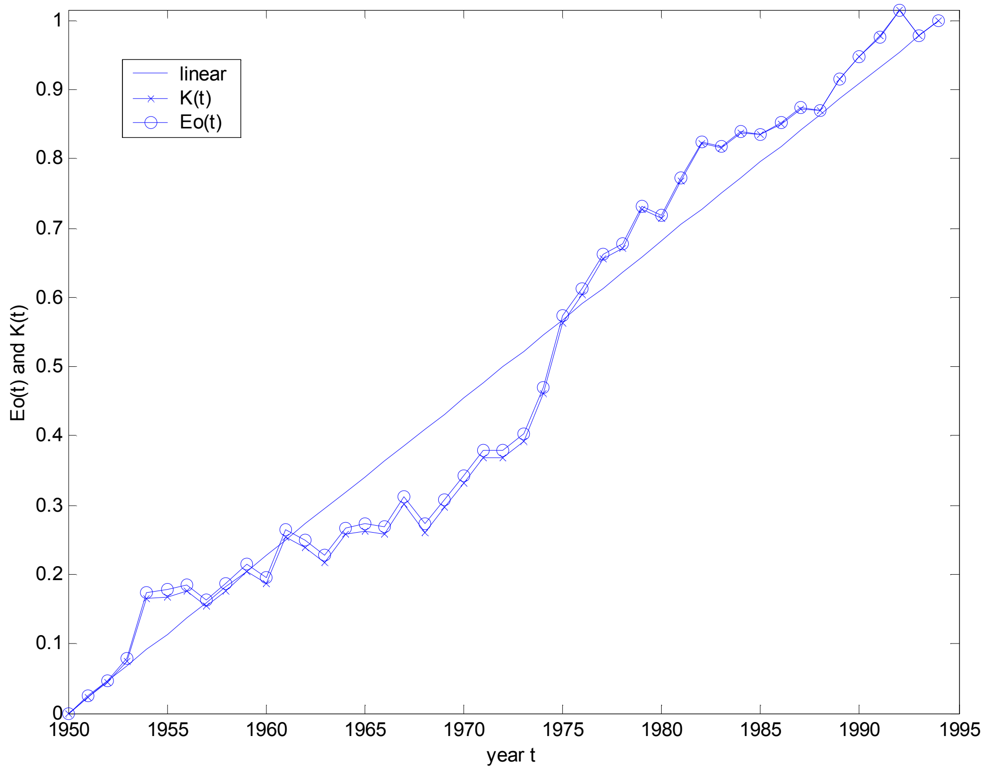


Figure 8. Values of b(x) for the G7 countries